Math 217 Fall 2025
Quiz 30 – Solutions

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1. Complete* the partial sentences below into precise definitions for, or precise mathematical characterizations of, the italicized term:

   (a) An *elementary matrix* is ...

   > **Solution:** An $n \times n$ matrix obtained by performing a *single* elementary row operation on the identity matrix I_n.

   (b) An *inner product* on a vector space V is ...

   > **Solution:** (Over $\mathbb{R}$.) A function $\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$ such that for all $\vec{u}, \vec{v}, \vec{w} \in V$ and all $a, b \in \mathbb{R}$:
   >
   > - **Bilinearity:**
   > $$\langle a\vec{u} + b\vec{v},\, \vec{w} \rangle = a\langle \vec{u}, \vec{w} \rangle + b\langle \vec{v}, \vec{w} \rangle, \quad \langle \vec{u},\, a\vec{v} + b\vec{w} \rangle = a\langle \vec{u}, \vec{v} \rangle + b\langle \vec{u}, \vec{w} \rangle;$$
   > - **Symmetry:** $\langle \vec{u}, \vec{v} \rangle = \langle \vec{v}, \vec{u} \rangle$;
   > - **Positive–definiteness:** $\langle \vec{v}, \vec{v} \rangle \geq 0$ for all $\vec{v}$, and $\langle \vec{v}, \vec{v} \rangle = 0$ if and only if $\vec{v} = \vec{0}$.

   (c) An *inner product space* is ...

   > **Solution:** A vector space V together with a specified inner product $\langle \cdot, \cdot \rangle$ on V. We denote it by $(V, \langle \cdot, \cdot \rangle)$.

2. Prove that if $\mathcal{U} = (\vec{u}_1, \ldots, \vec{u}_n)$ is an orthonormal basis of the inner product space V, then

$$\vec{x} = \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i \rangle \, \vec{u}_i$$

for all $\vec{x} \in V$.

> **Solution:** Let
> $$\vec{y} := \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i \rangle \, \vec{u}_i$$
> and consider the difference $\vec{z} := \vec{x} - \vec{y}$.

*For full credit, please write out fully what you mean instead of using shorthand phrases.

We claim that $\vec{z}$ is orthogonal to every basis vector $\vec{u}_k$. For any $k \in \{1, \dots, n\}$,

$$\begin{aligned}
\langle \vec{z}, \vec{u}_k \rangle &= \langle \vec{x} - \vec{y}, \vec{u}_k \rangle \\
&= \langle \vec{x}, \vec{u}_k \rangle - \left\langle \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i \rangle \, \vec{u}_i, \ \vec{u}_k \right\rangle \\
&= \langle \vec{x}, \vec{u}_k \rangle - \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i \rangle \, \langle \vec{u}_i, \vec{u}_k \rangle.
\end{aligned}$$

Since $\mathcal{U}$ is orthonormal, $\langle \vec{u}_i, \vec{u}_k \rangle = \delta_{ik}$, so the sum simplifies to

$$\sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i \rangle \, \langle \vec{u}_i, \vec{u}_k \rangle = \langle \vec{x}, \vec{u}_k \rangle.$$

Hence $\langle \vec{z}, \vec{u}_k \rangle = \langle \vec{x}, \vec{u}_k \rangle - \langle \vec{x}, \vec{u}_k \rangle = 0$ for all k.

Thus $\vec{z}$ is orthogonal to each vector in the orthonormal basis $\{\vec{u}_1, \dots, \vec{u}_n\}$. But these basis vectors span V, so the only vector orthogonal to all of them is the zero vector. Therefore $\vec{z} = \vec{0}$, which means

$$\vec{x} = \vec{y} = \sum_{i=1}^{n} \langle \vec{x}, \vec{u}_i \rangle \, \vec{u}_i,$$

as required.

3. True or False. If you answer true, then state TRUE. If you answer false, then state FALSE. Justify your answer with either a short proof or an explicit counterexample.

   (a) Every finite dimensional inner product space $(V, \langle \ , \ \rangle)$ has an orthonormal basis.

   Solution: TRUE.
   Let $\dim V = n$. Choose any basis $(\vec{v}_1, \dots, \vec{v}_n)$ of V. Applying the Gram–Schmidt process to this basis with respect to the given inner product $\langle \ , \ \rangle$ produces an orthonormal list $(\vec{u}_1, \dots, \vec{u}_n)$ that still spans V and is linearly independent. Thus $(\vec{u}_1, \dots, \vec{u}_n)$ is an orthonormal basis of V.

   (b) Suppose V is a vector space and $\{v_1, v_2, v_3, v_4\}$ is a set of vectors that span V. If $w \neq 0$ is another vector in V, then we can find $j \in \{1, 2, 3, 4\}$ such that the set

   $$\{w, v_k \mid k \in \{1, 2, 3, 4\} \setminus \{j\}\}$$

   spans V.

   Solution: TRUE.
   Since $\{v_1, v_2, v_3, v_4\}$ spans V, the dimension of V is at most 4, so V is finite dimensional. Consider the set $\{w, v_1, v_2, v_3, v_4\}$ of five vectors in V. Because $\dim V \leq 4$, this set is linearly dependent. Hence there exist scalars a, b_1, b_2, b_3, b_4, not all zero, such that

   $$aw + b_1 v_1 + b_2 v_2 + b_3 v_3 + b_4 v_4 = 0.$$

Since $w \neq 0$, we must have $a \neq 0$ (otherwise we would get a nontrivial linear dependence among $v_1, \dots, v_4$, which we do not need but can ignore). Thus

$$w = -\frac{b_1}{a}v_1 - \frac{b_2}{a}v_2 - \frac{b_3}{a}v_3 - \frac{b_4}{a}v_4.$$

At least one of the coefficients b_j is nonzero. Choose such a j. Then we can solve for v_j:

$$v_j = -\frac{a}{b_j}w - \sum_{k \neq j} \frac{b_k}{b_j}v_k.$$

This shows that v_j is in the span of $\{w\} \cup \{v_k : k \neq j\}$. Since $\{v_1, v_2, v_3, v_4\}$ spans V, replacing v_j with w leaves us with another spanning set:

$$\{w, v_k \mid k \in \{1,2,3,4\} \setminus \{j\}\} \text{ spans } V.$$

(c) Every finite dimensional vector space has an orthonormal basis.

Solution: FALSE.

The notion of "orthonormal" requires an *inner product*. A general finite dimensional vector space is not assumed to come equipped with any inner product at all. Without an inner product, the terms "orthogonal," "length," and hence "orthonormal basis" are not even defined.

For example, let V be any 2–dimensional real vector space considered purely as a vector space, with no inner product specified. Then it does *not* make sense to talk about an orthonormal basis of V; so the statement "every finite dimensional vector space has an orthonormal basis" is false as written.

(What *is* true is that every finite dimensional *inner product space* has an orthonormal basis, as in part (a).)